Progress on the theory wishlist for multiloop integrals

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Feynman integrals as iterated integrals (1)

- At one loop, only logarithm and dilogarithm needed

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2}$$

  - what functions will appear at higher loops?

- Logarithm and dilogarithm are first examples of iterated integrals with special ``d-log`` integration kernels

$$\frac{dt}{t} = d\log t \qquad \frac{-dt}{1-t} = d\log(1-t) \qquad \frac{dt}{1+t} = d\log(1+t)$$

- these are called harmonic polylogarithms (HPL) [Remiddi, Vermaseren]

  e.g. $H_{1,-1}(x) = \int_0^x \frac{dx_1}{1-x_1} \int_0^{x_1} \frac{dx_2}{1+x_2}$

  - shuffle product algebra
  - coproduct structure
  - Mathematica implementation [Maitre]

- weight: number of integrations

Feynman integrals as iterated integrals (2)

- Natural generalization: multiple polylogarithms

  [also called hyperlogarithms; Goncharov polylogarithms]

  allow kernels $w = d\log(t-a)$

  $$G_{a_1,\ldots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

  numerical evaluation: GINAC [Vollinga, Weinzierl]

- Chen iterated integrals

  $$\int_C \omega_1 \omega_2 \ldots \omega_n \qquad C : [0,1] \longrightarrow M$$ (space of kinematical variables)

  Alphabet: set of differential forms $\omega_i = d\log\alpha_i$

  integrals we discuss will be monodromy invariant on $M \setminus S$

  S (set of singularities)

  more flexible than multiple polylogarithms!

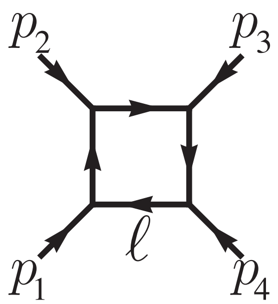
- Uniform weight functions (pure functions):

  $\mathbb{Q}$-linear combinations of functions of the same weight

d-log representations

- Can we make it manifest when integrals evaluate to pure functions?

$\mathcal{A}_4^{\ell=0} \times$



$$= \mathcal{A}_4^{\ell=0} \times \int \frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$

[Arkani-Hamed, Bourjaily, Cachzo, Goncharov, Postnikov, Trnka, 2012]

[Caron-Huot, talk at Trento, 2012]

[Lipstein, Mason, 2013]

$$\frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$
$$= \mathrm{dlog}\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

very suggestive! New ways of performing loop integrations?

Cuts and integrated integrands

- use cuts of integrals as guiding principle for finding convenient integral basis

[JMH, PRL 110 (2013) 25]

- integrals with simple cuts are expected to integrate to uniform weight functions

idea: any cut that completely localizes the integral should give just a rational number

Strategy for computing Feynman integrals using differential equations

- Useful facts:

  (1) For a given problem, one can choose
  a finite basis of Feynman integrals

  (2) Basis integrals satisfy coupled
  first-order differential equations

  (3) many classes of Feynman integrals
  evaluate to iterated integrals

- Idea: choose basis such that the differential equations are simple, and such that (3) is made obvious

Key points of the method

[JMH, PRL 110 (2013) 25]

- differential equations for master integrals $\vec{f}$
- crucial: choose convenient basis (systematic procedure)
  $\longrightarrow$ makes solution trivial to obtain
- elegant description: Feynman integrals specified by:
  (1) set of 'letters' (related to singularities x_k)
  (2) set of constant matrices A_k

Example: one dimensionless variable x; $D = 4 - 2\epsilon$

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x;\epsilon)$$

- expansion to any order in ϵ is linear algebra

answer: multiple polylogarithms of uniform weight ('transcendentality')

- asymptotic behavior $\vec{f}(x;\epsilon) \sim (x - x_k)^{\epsilon A_k} \vec{f}_0(\epsilon)$
- natural extension to multi-variable case

Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k \underbrace{A_k}_{\text{constant matrices}} \log \underbrace{\alpha_k(\vec{x})}_{\text{letters (alphabet)}}\right] \vec{f}(\vec{x};\epsilon)$$

- Examples of alphabets:

4-point on-shell $\qquad \alpha = \{x, 1+x\}$

two-variable example (from 1-loop Bhabha scattering): $\qquad \alpha = \{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

more complicated examples later

> - Matrices and letters determine solution
> - Immediate to solve in terms of Chen iterated integrals

Important points differential equations

• Uniform weight basis can be found systematically using cuts (related to d-log representations) [Arkani-Hamed et al.] [J.M.H.]

other ideas [Mastrolia et al.] [Caron-Huot, J.M.H.] [Gehrmann et al.]

• DE provide information about integrals in compact form (alphabet, matrices)

• contain more information than epsilon expansion: exact limits

• boundary conditions often for free (e.g. finiteness in certain limits)

application: bootstrap for single-scale integrals [J.M.H., A.V. Smirnov, V.A.Smirnov]

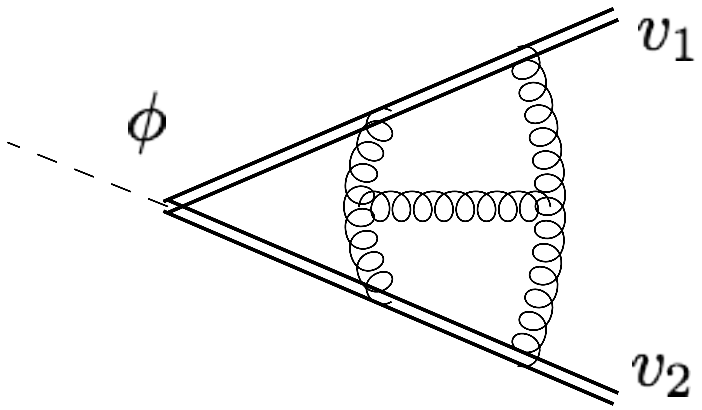
• Chen iterated integrals give most compact form of answer

• To given weight, answer can be rewritten in terms of minimal function basis [Goncharov]

3-loop HQET integrals

- 8 integral families, e.g.

$$\cos\phi = \frac{v_1 \cdot v_2}{\sqrt{v_1^2}\sqrt{v_2^2}}\,, \quad x = e^{i\phi}$$



- alphabet $\alpha = \{x, 1+x, 1-x\}$

- 71 master integrals

- application: QCD cusp anomalous dimension

[Grozin, J.M.H.,Korchemsky, Marquard, to appear 2014]

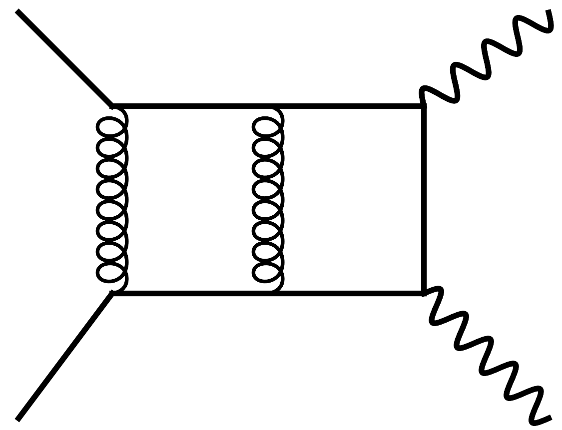
physics motivation:
infrared divergences of massive scattering amplitudes

Vector boson production integrals $pp \to VV'$

[JMH, Melnikov, V. Smirnov, JHEP 1430 (2014)]

[JMH, Caola, Melnikov, V. Smirnov, 1404.5590]

- sample integral family

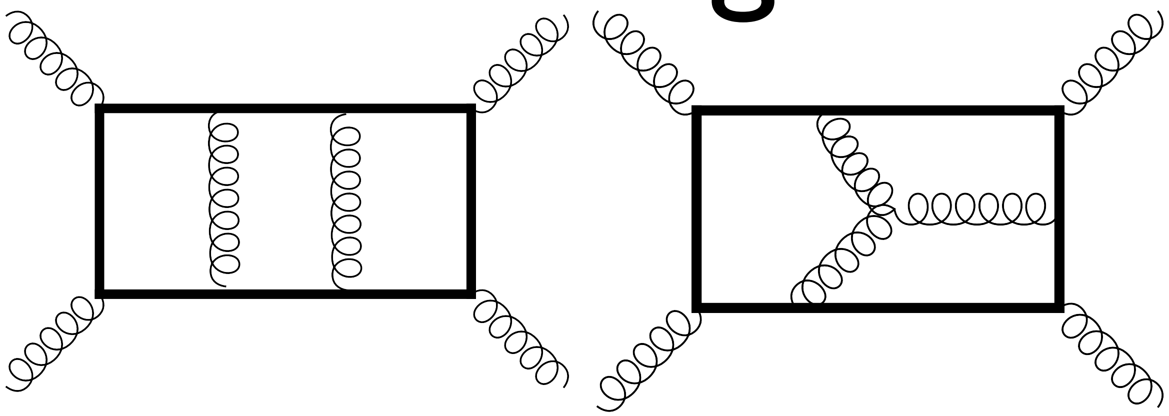


- variables S, T, M_3^2, M_4^2
- parametrization $\frac{S}{M_3^2} = (1+x)(1+xy)\,, \quad \frac{T}{M_3^2} = -xz\,, \quad \frac{M_4^2}{M_3^2} = x^2 y$
- physical region $0 < x\,, \qquad 0 < y < z < 1$
- alphabet

$$\begin{aligned}\alpha =&\{x, y, z, 1+x, 1-y, 1-z, 1+xy, z-y, 1+y(1+x)-z, xy+z,\\ &1+x(1+y-z), 1+xz, 1+y-z, z+x(z-y)+xyz, z-y+yz+xyz\}.\end{aligned}$$

- boundary condition computed at $x \to 0\,, \quad y \to z \to 1$

Massive integrals for light-by-light scattering



[Caron-Huot, J.M.H., 2014]

- variables m^2, s, t

3 loops and 3 scales!

- full set of 2-loop master integrals (at 3 loops: all finite master integrals in D=4)

similar integrals in QCD, e.g. for finite top quark mass

- alphabet

$$\alpha = \Big\{ u, 1+u, v, 1+v, u+v, \frac{\beta_u - 1}{\beta_u + 1}, \frac{\beta_v - 1}{\beta_v + 1}, \frac{\beta_{uv} - 1}{\beta_{uv} + 1}, \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u}, \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v},$$

$$u^2 - 4v, v^2 - 4u, \frac{2 - 2\beta_{uv} + u}{2 + 2\beta_{uv} + u}, \frac{2 - 2\beta_{uv} + v}{2 + 2\beta_{uv} + v},$$

$$1 + u + v\,, \frac{4 - v + \beta}{4 - v - \beta}\,, \frac{4 + v + \beta}{4 + v - \beta}\,, \frac{(4\beta_u + \beta)(4\beta_u + \beta_u v + \beta)}{(4\beta_u - \beta)(4\beta_u + \beta_u v - \beta)}\,, \frac{(4\beta_{uv} + \beta)(4\beta_{uv} - \beta_{uv} v + \beta)}{(4\beta_{uv} - \beta)(4\beta_{uv} - \beta_{uv} v - \beta)} \Big\}$$

where $u = -4m^2/s\,, \quad v = -4m^2/t\,,$

$$\beta_u = \sqrt{1+u}\,, \quad \beta_v = \sqrt{1+v}\,, \qquad \beta_{uv} = \sqrt{1+u+v}\,, \qquad \beta = \sqrt{16 + 16u + 8v + v^2}$$

- efficient numerical representation for Chen iterated integrals

$$g_{37}(2,4) = \quad 0.07649227172719869702548592574 68\ldots$$

Factory-line for master integrals

Workflow (typical time)

integral reduction 1-3

choose canonical basis for differential equations 1-2

analyze alphabet 0-2

boundary condition (analytic continuation) 1-5

ϵ expansion 0

different representations 1-5

communicate results arXiv:... 30

More efficient way to communicate results-
Online library for loop integrals?

key data: integral basis,
matrices, letters A_k , α_k

Thank you!

Extra slides

A word of caution: more exotic objects

- mathematicians like to consider single-scale Feynman integrals

- conjecture that certain periods only evaluate to multiple zeta values (MZV) appear disproven by [Brown, Schnetz]

- Elliptic functions

relevant e.g. in top quark physics Czakon et al.

also appear in massless N=4 SYM [Caron-Huot, Larsen]

recent work Elliptic polylogairthms [Brown, Levin]

[Bloch, Vanhove] [Vanhove] [Remiddi, Tancredi] [Adams, Bogner, Weinzierl]

Note: weight property generalizes weight n -> (n/2,n/2) mixed Hodge theory

systematic and practical way for dealing with them for practical applications?

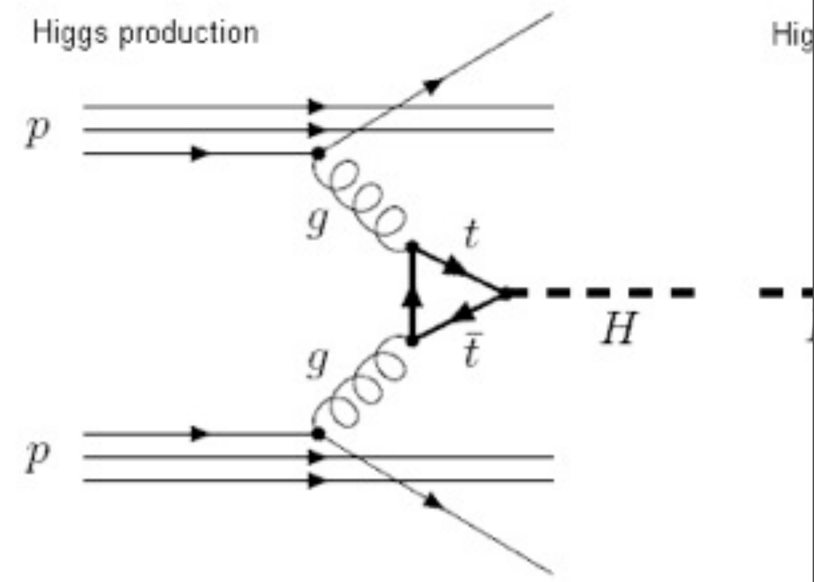
- Here: cases where Chen iterated integrals are sufficient

The alphabet and perfect bricks (1)

Can we parametrize variables such that alphabet is rational?

Not essential, but nice feature.

- Example: Higgs production



encounter $\sqrt{1 - 4m^2/s}$

choose $-m^2/s = x/(1-x)^2$

$\alpha = \{x, 1-x, 1+x\}$ (to two loops)

Note: this is a purely kinematical question. Independent of basis choice.

- Related to diophantine equations

e.g. find rational solutions to equations such as

$$1 + 4\,a = b^2$$

here we found a 1-parameter solution

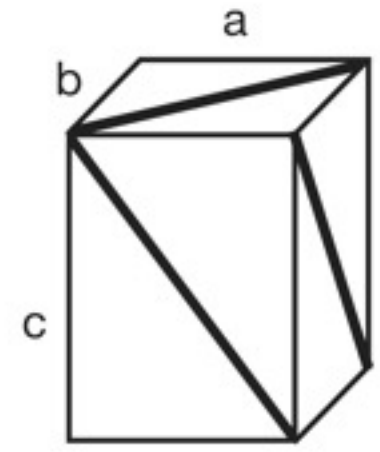
$$a = \frac{x}{(1-x)^2} \qquad b = \frac{1+x}{1-x}$$

The alphabet and perfect bricks (2)

- Classic example: Euler brick problem

Find a brick with sides a, b, c
and diagonals d, e, f integers

$$a^2 + b^2 = d^2\,,$$
$$a^2 + c^2 = e^2\,,$$
$$b^2 + c^2 = f^2\,.$$

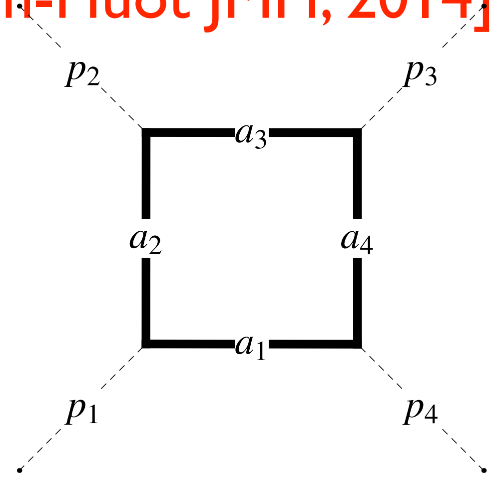


smallest solution (P. Halcke):
(a,b,c)=(44,117,240)

Perfect cuboid (add eq. $a^2 + b^2 + c^2 = g^2$): open problem in mathematics!

- Similar equations for particle kinematics

[Caron-Huot JMH, 2014]



e.g encountered in 4-d light-by-light scattering

$$u = -4m^2/s \qquad v = -4m^2/t$$

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

Need two-parameter solution to

$$\beta_u^2 + \beta_v^2 = \beta_{uv}^2 + 1$$

e.g. $$\beta_u = \frac{1-wz}{w-z}\,,\quad \beta_v = \frac{w+z}{w-z}\,,\quad \beta_{uv} = \frac{1+wz}{w-z}\,.$$

more roots in D-dim and at 3 loops! - in general alphabet changes with the loop order!

Find such solutions systematically? Minimal polynomial order?

Goncharov weight four conjecture

- rewrite any multiple polylogarithm in terms of function basis [Goncharov]

  e.g. at weight 4 (important for NNLO computations)

$$\{\log(x)\log(y)\log(z)\log(w), \log(x)\log(y)\mathrm{Li}_2(z), \\ \mathrm{Li}_2(x)\mathrm{Li}_2(y), \log(x)\mathrm{Li}_3(y), \mathrm{Li}_4(x), \mathrm{Li}_{2,2}(x,y)\}$$

  for set of arguments (to be found - symbol/coproduct provides guidance)

  minimal set of integration kernels vs. minimal set of function arguments

- practical tool: ``symbol`` useful projections [Goncharov, Spradlin, Vergu, Volovich]

  [Brown] [Goncharov] [Duhr, Gangl, Rhodes]

  e.g. project on $\mathrm{Li}_{2,2}(x,y)$ part

  e.g. project out all products

  lecture notes: [Vergu]
  [Brown][Zhao]

- ``symbol`` = Chen iterated integral without boundary information

  diff. eqs. or other information can be used to fix this

Equivalent representations

- version 1: Chen iterated integrals

[most compact]

[flexible: analytic continuation, limits]

$$g_6 = \int_\gamma d\,\log \frac{\beta_u - 1}{\beta_u + 1}\, d\log \frac{\beta_{uv} - \beta_u}{\beta_{uv} + \beta_u} + \int_\gamma d\,\log \frac{\beta_v - 1}{\beta_v + 1}\, d\,\log \frac{\beta_{uv} - \beta_v}{\beta_{uv} + \beta_v}\,.$$

[easy to see DE, cuts]

[2 loops: 10 terms]

[ideas for numerics: J.M.H., Caron-Huot]

- version 2: Goncharov polylogarithms

(if alphabet rational in at least one variable)

[longer expressions; requires rational alphabet; GINAC numerical evaluation]

$$g_6 = -G_{-1,0}(w) + G_{0,-1}(w) - G_{0,1}(w) + G_{1,0}(w) + H_{-1,0}(z) - H_{0,-1}(z) - H_{0,1}(z) + H_{1,0}(z) - G_0(w)H_{-1}(z) + G_{-1}(w)H_0(z) - G_1(w)H_0(z) - G_0(w)H_1(z)\,.$$

[2 loops: 2-3 pages]

- version 3: minimal function basis $g_6 = -\beta_{uv}/2I_1$

$$I_1 = \frac{2}{\beta_{uv}}\left\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2} + \sum_{i=1,2}\left[2\,\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\right]\right\}.$$

[arbitraryness; usually long expressions; good at low weight; fast numerical evaluation]

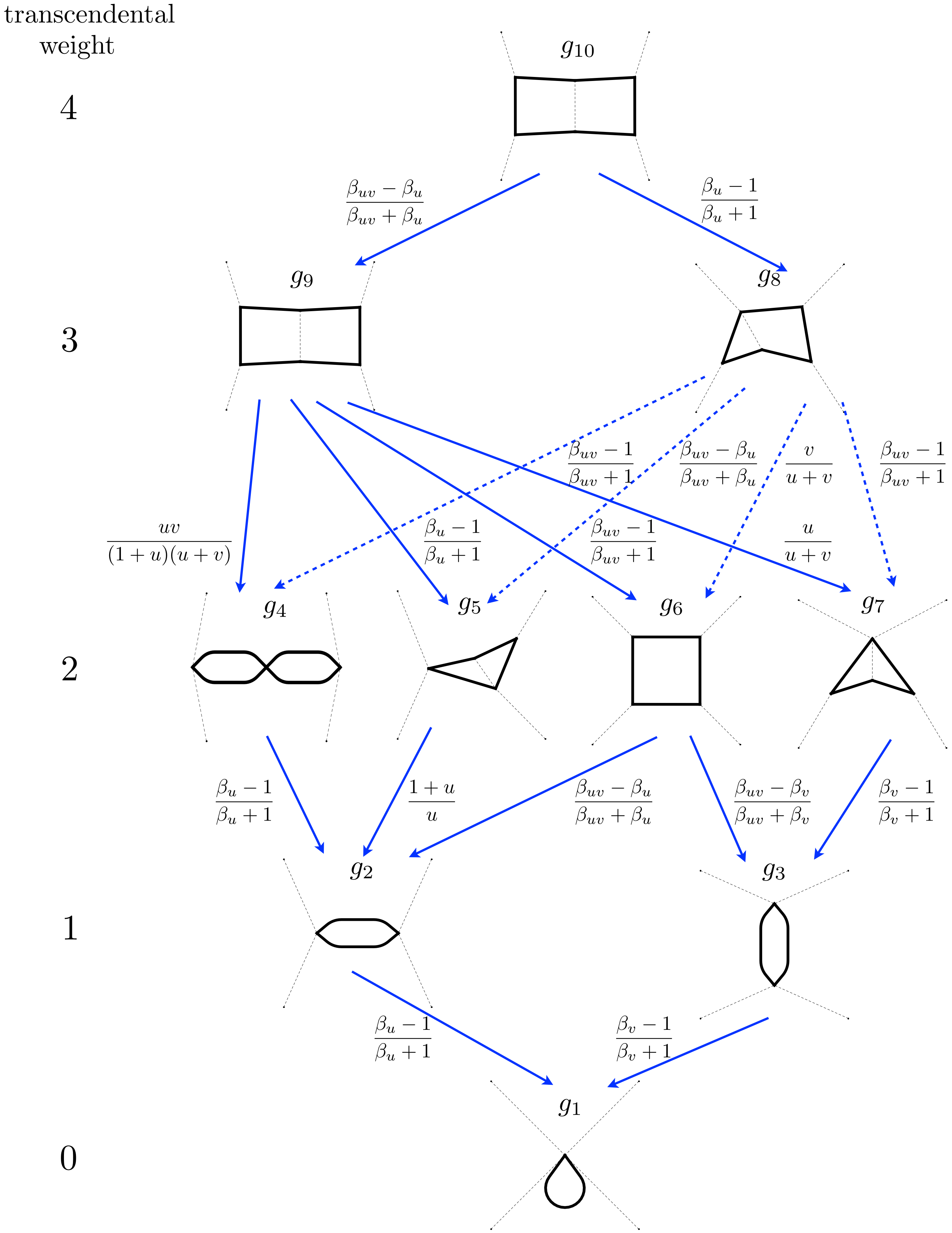
[2 loops: several pages]

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

- some examples from literature: [Goncharov et al.] [Duhr] [Gehrmann et al.] ...

Iterative structure for finite loop integrals

[Caron-Huot, J.M.H. (2014)]



- block triangular matrix structure (weight grading)
- algorithm for finding this form